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Symmetries of Physical Theories

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A physical theory is, by definition, a complete orthomodular atomic lattice having the covering property. Given L a quantum logic and S_L the set of all its states, a theorem is proved which asserts that, if certain reasonable assumptions concerning S_L are satisfied, then for any bijective convex mapping U: $S_L \to S_L$, satisfying also certain physically meaningful conditions, there exists a unique automorphism V: $L \to L$ such that $U(p) = p \circ V^{-1}$ for all $p \in S_L$.

1. INTRODUCTION

In this work complete orthomodular atomic lattices having the covering property (COMALC) will be called physical theories. Given L a physical theory, we will denote by S_L the set of all its states and by $\mathcal{O}_L$ the set of all its observables. In our language, any observable of L is a Boolean subalgebra of L (Ivanov, 1992).

We intend to prove a theorem concerning automorphism of L, which will be formulated in the next paragraph. Since this theorem is strongly connected with the symmetries of physical theories, we will discuss first the general problem of symmetries. It will be easily seen that this discussion gives also a quite transparent physical interpretation of some mathematical conditions required for proving the above-mentioned theorem.

One of the most general definitions of a physical theory is the following: a physical theory is a pair $T = (\mathcal{O}, S)$ of two nonempty sets, which are called the set of observables and the set of states of the theory T. For the sake of convenience we may consider that the set of all possible values of any given observable $\omega \in \mathcal{O}$ is a subset of R. In this case the result of the measurement of a given observable ω in a given state σ is considered to be a probability $P(\omega, \sigma)$: $\mathcal{B}(R) \to [0, 1]$, where $\mathcal{B}(R)$ is the set of all Borel subsets of the

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real line R (Holevo, 1982). It is quite clear that $P(\omega, \sigma)$ are those objects determined by a theory which have a direct empirical significance. Therefore, it is natural to use the set $\{P(\omega, \sigma); \omega \in \mathbb{O}, \sigma \in S\}$ for defining the symmetries of the theory T.

Definition 1. A pair (U, V) of bijective mappings, $U: \mathbb{O} \to \mathbb{O}$, $V: S \to S$, is said to be a symmetry of T if for all pairs (ω, σ) we have

$$P(U(\omega), V(\sigma)) = P(\omega, \sigma) \tag{A1'}$$

Now let L be a COMALC. It is clear that L defines naturally a physical theory, which is the pair $(\mathbb{O}_L, S_L)$. It is also obvious that, given $\omega \in \mathbb{O}_L$, $\sigma \in S_L$, the probability $P(\omega, \sigma)$ is essentially represented by the restriction of the states σ to the Boolean algebra $\omega \subseteq L$ (Ivanov, 1992). We want to get another form of the condition (A1′), which refers to the theory $(\mathbb{O}_L, S_L)$. We know that the elements $a \in L$ are usually interpreted as tests ("yes–no" experiments). Given $a \in L$ a test, we may define the observable $\omega_a = \{0, 1, a, a^{\perp}\}$, where 0 and 1 are the lowest and the greatest element of L, respectively, and $a^{\perp}$ the orthocomplement of a. It is easy to understand that a and ω_a are, in fact, physically identical objects, so that we might say that ω_a is a test-type observable. Taking account of the significance of symmetry, it is clear that $U(\omega_a)$ must be also a test-type observable, defined by a test which will be denoted by $U(a)$. Now it is easy to check that $U: L \to L$ is a bijective mapping and it follows that, in the case of the theory $(\mathbb{O}_L, S_L)$, the condition (A1′) becomes

$$[V(\sigma)](U(a)) = \sigma(a) \tag{A1}$$

for all $a \in L$. It may also be shown that V is a convex mapping. In order to do this, let us consider the state $\sigma = \sum_{i=1}^n c_i\sigma_i$, where $\sigma_i \in S_L$ and $c_i > 0$, $\sum_{i=1}^n c_i = 1$. We have to prove that

$$V(\sigma) = \sum_{i=1}^{n} c_i V(\sigma_i) \tag{A2}$$

Taking account of (A1), we may write

$$[V(\sigma)](U(a)) = \sigma(a) = \sum_{i=1}^{n} c_i\sigma_i(a)$$

and

$$\left[\sum_{i=1}^{n} c_i V(\sigma_i)\right](U(a)) = \sum_{i=1}^{n} c_i[V(\sigma_i)](U(a)) = \sum_{i=1}^{n} c_i\sigma_i(a)$$

We get

$$[V(\sigma)](U(a)) = \left[\sum_{i=1}^{n} c_i V(\sigma_i)\right](U(a))$$

for all $a \in L$ and (A2) is proved since U is a bijective mapping.

The conclusion of this discussion is that, in the case of a theory $(\mathcal{O}_L, S_L)$, any symmetry is defined by a pair (V, U) of bijective mapping, V: $S_L \to S_L$, U: $L \to L$, V being a convex mapping. In this paper we will prove, roughly speaking, that, given V: $S_L \to' S_L$ a convex mapping, then, provided the condition (A1) is satisfied, U is an automorphism of L. To be more precise, we will show that, given V: $S_L \to S_L$ a bijective convex mapping, there exists a unique automorphism $\mathcal{V}$: $L \to L$ such that $V(p) = p \circ \mathcal{V}^{-1}$ for all $p \in S_L$. This means that, if a physical theory is identified with a COMALC L, then all its symmetries are described by automorphism of L.

2. SYMMETRIES AND AUTOMORPHISMS OF PHYSICAL THEORIES

Let L be a physical theory and $\Omega(L)$ the set of all its atoms. We will denote by $\leq$ the order relation on L and by $\vee$, $\wedge$, and $\perp$ the join, the meet, and the orthocomplementation on L, respectively. The notation $(a, b)\perp$ means that the elements a, b are orthogonal.

In order to prove our result we have to make the following assumptions concerning the set S_L:

(S1) For any atom $\alpha \in \Omega(L)$ there exists $p \in S_L$ such that $p(\alpha) = 1$.
(S2) $\alpha \in \Omega(L)$, $p_1, p_2 \in S_L$, $p_1(\alpha) = p_2(\alpha) = 1 \Rightarrow p_1 = p_2$.
(S3) $\alpha, \beta \in \Omega(L)$, $p \in S_L$, $p(\alpha) = p(\beta) = 1 \Rightarrow \alpha = \beta$.

If L is a classical theory, i.e., an atomic Boolean algebra, then the conditions (S1)–(S3) are trivially satisfied. On the other hand, any theory may be considered as a union of its classical components, which are maximal atomic Boolean subalgebras of L (Ivanov, 1992). Since the conditions (S1)–(S3) are satisfied by each classical component of L, it is natural—at least from the physical point of view—to admit that they are satisfied also by L.

The unique state taking the value 1 on the atom will be denoted by δ_α. Such states will be called occasionally δ-states. It is easy to prove that any δ-state is a pure state. It seems that the converse of this statement cannot be proved in this framework without supplementary assumptions, so that we will consider also the following condition:

(S4) Any pure state on L is a δ-state.

This assumption ends the list of conditions imposed on S_L which are used for proving the theorem mentioned in the Introduction. In order to state and prove this theorem we need some technical results which will be presented as a sequence of propositions.

We begin with a lemma referring to a special property of δ-states.

Lemma. Let L be a quantum logic and $\alpha \in \Omega(L)$. Then for any $a \in L$, $\alpha > 0$, there exists an atom $\gamma \leq a$ such that $\delta_\alpha(a) = \delta_\alpha(\gamma)$.

Proof. There is nothing to prove when $a \in \Omega(L)$, $\alpha \leq a$, or $(\alpha, a)\perp$. Therefore, let us suppose that $\alpha \not\leq a$ and $(\alpha, a)\not\perp$. Since L has the covering property, there exists an atom β, $(\beta, a)\perp$ such that $a \vee \alpha = a \vee \beta$. We have $\delta_\alpha(a \vee \alpha) = \delta_\alpha(a) + \delta_\alpha(\beta) = 1$. On the other hand, there exists $\gamma \in \Omega(L)$, $(\gamma, \beta)\perp$ such that $\alpha \vee \beta = \gamma \vee \beta$. Since $(\alpha \vee \beta) = \delta_\alpha(\gamma) + \delta_\alpha(\beta) = 1$, we get easily the equality $\delta_\alpha(\gamma) = \delta_\alpha(a)$. It remains to prove that $\gamma \leq a$. But $\gamma \leq \alpha \vee \beta \leq a \vee \alpha = a \vee \beta$ and $(\gamma, \beta)\perp$ imply $\gamma \leq a$ since $\gamma \leq (a \vee \beta) \wedge \beta^\perp = a$.

Proposition 1. Let U: $S_L \to S_L$ be a convex bijective mapping. Then there exists an unique bijective mapping V: $\Omega(L) \to \Omega(L)$ such that

$$U(\delta_\alpha) = \delta_{V(\alpha)}, \qquad U^{-1}(\delta_\alpha) = \delta_{V^{-1}(\alpha)}$$

Proof. It is easy to see that $U(\delta_\alpha)$ is a pure state. Indeed, if $U(\delta_\alpha) = \nu_1 p_1 + \nu_2 p_2$, ν_1, $\nu_2 > 0$, $\nu_1 + \nu_2 = 1$, $p_1 \neq p_2$, then, since U^{-1} is also convex, $\delta_\alpha = \nu_1 U^{-1}(p_1) + \nu_2 U^{-1}(p_2)$, which means that δ_α is not a pure state. Since $U(\delta_\alpha)$ is a pure state, we may find a *unique* $\beta \in \Omega(L)$ such that $U(\delta_\alpha) = \delta_\beta$. Therefore, we may define a mapping V: $\Omega(L) \to \Omega(L)$ by the equality $\beta = V(\alpha)$, where β is obviously that atom which satisfies the equality $U(\delta_\alpha) = \delta_\beta$.

Suppose now that $\alpha \neq \beta$, α, $\beta \in \Omega(L)$. From (S3) we know that $\delta_\alpha \neq \delta_\beta$. It follows that $U(\delta_\alpha) \neq U(\delta_\alpha) \Rightarrow \delta_{V(\alpha)} \neq \delta_{V(\beta)} \Rightarrow V(\alpha) \neq V(\beta)$, so that V is injective. Let now β be an atom of L. Obviously, $U^{-1}(\delta_\beta)$ is a δ-state, so that $U^{-1}(\delta_\beta) = \delta_\alpha$, $\alpha \in \Omega(L)$. It results that $\delta_\beta = U(\delta_\alpha) = \delta_{V(\alpha)}$, $\beta = V(\alpha)$, and V is also surjective. The equality $U^{-1}(\delta_\alpha) = \delta_{V^{-1}(\alpha)}$ results easily from the following chain of implication:

$$U^{-1}(\delta_\alpha) = \delta_\beta \Rightarrow \delta_\alpha = U(\delta_\beta) \Rightarrow V(\beta) = \alpha \Rightarrow \beta = V^{-1}(\alpha) \Rightarrow U^{-1}(\delta_\alpha) = \delta_{V^{-1}(\alpha)}$$

Taking account of the completeness of L, we may define the following two mappings:

$$\mathcal{V}: L \to L\, \mathcal{V} = \bigvee_{\substack{\alpha \in \Omega(L) \\ \alpha \leq a}} V(\alpha)$$

$$\mathcal{U}: L \to L\, \mathcal{U} = \bigvee_{\substack{\alpha \in \Omega(L) \\ \alpha \leq a}} V^{-1}(\alpha)$$

We will assume that the following two conditions, obviously inspired by (A1), are satisfied for all $p \in S_L$, $a \in L$:

$$[U(p)](\mathcal{V}(a)) = p(a) \tag{Ai}$$

$$[U^{-1}(p)](\mathcal{U}(a)) = p(a) \tag{Aii}$$

We will consider also for any $a \in L$ the set $P(a) = \{p \in S_L; p(a) = 1\}$.

Proposition 2. The following two equalities hold for any convex mapping U which satisfies (Ai), (Aii):

$$P(\mathcal{V}(a)) = U(P(a)) \tag{i}$$

$$P(\mathcal{U}(a)) = U^{-1}(P(a)) \tag{ii}$$

Proof. It is sufficient to prove (i). Let us take $p \in P(\mathcal{V}(a))$. By definition, $p(\mathcal{V}(a)) = 1$. We have to find $p' \in P(a)$ such that $p = U(p')$. Let us consider the state $U^{-1}(p)$ and prove that $[U^{-1}(p)](a) = 1$. By applying (Ai) to the state $U^{-1}(p)$, we get $[U(U^{-1}(p))](\mathcal{V}(a)) = [U^{-1}(p)](a)$. Since the left-hand side of this equality equals $p(\mathcal{V}(a)) = 1$, we get $[U^{-1}(p)](a) = 1$, so that $p' = U^{-1}(p)$. Conversely, given $p \in U(P(a))$, there exists $p' \in P(a)$ such that $p = U(p')$. In this situation $p(\mathcal{V}(a)) = [U(p')](\mathcal{V}(a) = p'(a) = 1$ and the proof is complete.

Proposition 3. Let $a > 0$ be an element of L and $\beta \in \Omega(L)$, $\beta \leq \mathcal{V}(a)$. Then there exists an atom $\alpha \leq a$ such that $\beta = V(\alpha)$.

Proof. Let us assume that there exists an atom $\beta \leq \mathcal{V}(a)$ such that $\beta \neq V(\alpha)$ for all $\alpha \leq a$. Then, obviously, $V^{-1}(\beta) \nleq a$ and, consequently, $\delta_{V^{-1}(\beta)} \notin P(a)$ [this is because the lemma combined with the property (S3) makes true the implication $\delta_\alpha(a) = 1 \Rightarrow \gamma \leq a$]. On the other hand, $\delta_\beta \in P(\mathcal{V}(a))$ and $U^{-1}(\delta_\beta) = \delta_{V^{-1}(\beta)}$, so that $\delta_{V^{-1}(\beta)} \in U^{-1}(P(\mathcal{V}(a)))$. By using Proposition 2(i), we get $\delta_{V^{-1}(\beta)} \in P(a)$. The obtained contradiction proves our result.

Corollary. The mapping V defines a one-to-one correspondence between the sets $\Omega(a) = \{\alpha \in \Omega(L); \alpha \leq a\}$ and $\Omega(\mathcal{V}(a)) = \{\beta \in \Omega(L); \beta \leq \mathcal{V}(a)\}$.

Proposition 4. $\mathcal{U}$ and $\mathcal{V}$ are order-preserving mappings and $\mathcal{U} = \mathcal{V}^{-1}$.

Proof. Obviously $a \leq b \Rightarrow \mathcal{U}(a) \leq \mathcal{U}(b)$, $\mathcal{V}(a) \leq \mathcal{V}(b)$. Then, from the corollary we know that $\{V^{-1}(\beta); \beta \leq \mathcal{V}(a)\} = \{\alpha; \alpha \leq a\}$. It follows that

$$\mathcal{U}(\mathcal{V}(a)) = \vee\{V^{-1}(\beta); \beta \leq \mathcal{V}(a)\} = V\{\alpha; \alpha \leq a\} = a$$

Similarly, $\mathcal{V}(\mathcal{U}(a)) = a$ and the proof is complete.

Proposition 5. $V(a^{\perp}) = V(a)^{\perp}$.

Proof. We will show first that $(\alpha, \beta)\perp$, $\alpha, \beta \in \Omega(L)$, implies $(V(\alpha), V(\beta))\perp$. Indeed, if we put $p = \delta_\alpha$, $a = \beta$ in the equality (Ai), then we obtain easily $\delta_\alpha(\beta) = \delta_{V(\alpha)}(V(\beta)$. Since $(\alpha, \beta)\perp \Rightarrow \delta_\alpha(\beta) = 0$, we get $\delta_{V(\alpha)}(V(\beta)) = 0$, which implies, according the lemma, $(V(\alpha), V(\beta))\perp$. By using this fact, we may find with no difficulty $(a, b)\perp \Rightarrow (\mathcal{V}(a), \mathcal{V}(b))\perp$.

By applying this implication to the pair $(a, a^{\perp})$, we get $(\mathcal{V}(a), \mathcal{V}(a^{\perp}))\perp$ or $\mathcal{V}(a^{\perp}) \leq \mathcal{V}(a)^{\perp}$. Since L is orthomodular, we may find $b \in L$, $(b, \mathcal{V}(a^{\perp}))$ such that $\mathcal{V}(a)^{\perp} = b \vee \mathcal{V}(a^{\perp})$. Then, $b \leq \mathcal{V}(a^{\perp})^{\perp} \Rightarrow \mathcal{V}(a^{\perp}) \leq b^{\perp} \Rightarrow a^{\perp} \leq \mathcal{V}^{-1}(b^{\perp})$ and $b \leq \mathcal{V}(a)^{\perp} \Rightarrow \mathcal{V}(a) \leq b^{\perp} \Rightarrow a \leq \mathcal{V}^{-1}(b^{\perp})$. It follows that $1 = \mathcal{V}^{-1}(b^{\perp})$, then $b^{\perp} = 1$ and $b = 0$. The proof is complete.

Now we may state the main result of this work.

Theorem. Let U: $S_L \to S_L$ be a convex bijection satisfying (Ai)–(Aii). Then there exists a unique automorphism $\mathcal{V}$: $L \to L$ such that $U(p) = p \circ \mathcal{V}^{-1}$, for all $p \in S_L$.

Proof. In the previous series of propositions we proved the existence of an automorphism $\mathcal{V}$ having the property (Ai). Obviously, (Ai) may be also written in the form $U(p) \circ \mathcal{V} = p$, which gives immediately $U(p) = p \circ \mathcal{V}^{-1}$. Suppose now that there exists an automorphism χ: $L \to L$ such that $U(p) \circ \chi = p$ for all $p \in S_L$. Then $p \circ \mathcal{V}^{-1} = p \circ \chi^{-1}$ for all $p \in S_L$. Since $\mathcal{V}$ and χ are automorphisms, the equality $\mathcal{V} = \chi$ holds if $\mathcal{V}(\alpha) = \chi(\alpha)$ for all $\alpha \in \Omega(L)$. Given an arbitrarily fixed $\alpha \in \Omega(L)$, we may write $\delta_\alpha \circ (\mathcal{V}^{-1} \circ \chi) = \delta_\alpha$, so that $\delta_\alpha[(\mathcal{V}^{-1} \circ \chi)(\alpha)] = 1$ and $(\mathcal{V}^{-1} \circ \chi)(\alpha) = \alpha$. It follows that $\mathcal{V}(\alpha) = \chi(\alpha)$ for all $\alpha \in \Omega(L)$.

3. COMMENTS

The theorem proved in Section 2 is, in fact, an "algebraic analog" of a well-known theorem existing in the standard Hilbert-space quantum mechanics (Varadarajan, 1968). Of course, the proof of the traditional version of our theorem was obtained by using the powerful Hilbert-space techniques. Since we tried to prove this result by using only lattice-theoretic methods, we had to make several assumptions which are not necessary when the Hilbert-space techniques are used. We have in mind the conditions (S1)–(S4) and (Ai)–(Aii).

But, as has been seen, (S1)–(S4) have a reasonable motivation and (Ai)–(Aii) result directly from the general notion of symmetry. Therefore, even in the case that some of our assumptions might be proved, accepting the above-mentioned conditions as hypotheses is justified.

We want to end these short comments with some observations concerning the conditions which were used for proving our theorem.

The convexity of U and the condition (S4) were used only for constructing the mapping V: $\Omega(L) \to \Omega(L)$.

The completeness of L is necessary for extending naturally the mapping V to the lattice L by defining $\mathcal{V}$, so that it seems to be an indispensable condition.

Once $\mathcal{V}$ and $\mathcal{U}$ are defined, it is almost obvious that it is impossible to prove that they are automorphisms without additional assumptions. Probably (Ai) and (Aii) are among the most natural hypotheses.

The covering property is also an indispensable condition. Indeed, it was used for proving the lemma which was considered at several central points of our proof.

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